

A General Classification of Three-Neutrino Models and U_{e3}

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Abstract

A classification of models with three light neutrinos is given. This classification includes virtually all of the three-neutrino models proposed in the last few years, of which there are approximately one hundred. The essential ideas, attractive features, and characteristic problems of the different classes of model are discussed. The classification is based principally on how large $\nu_\mu - \nu_\tau$ mixing is obtained. A general discussion of the mixing parameter U_{e3} is then given, showing what values are to be expected for it in each type of model.

1 Introduction

Over the years, and especially since the discovery of the large mixing of ν_μ seen in atmospheric neutrino experiments, there have been numerous models of neutrino masses proposed in the literature. In the last two years alone, as many as one hundred different models have been published. One of the goals of this paper is to give a helpful classification of these models. Such a classification is possible because in actuality there are only a few basic ideas that underlie the vast majority of published neutrino mixing schemes. After some preliminaries, we present in section 2 a general classification of three-neutrino models that have a hierarchical neutrino spectrum. In section 3 we discuss the parameter U_{e3} , which describes the ‘1-3’ mixing of neutrinos. Since theoretical models are constructed to account for the solar and atmospheric neutrino oscillation data, which tightly constrain the ‘1-2’ and ‘2-3’ mixings but not the ‘1-3’ mixing, the parameter U_{e3} will be very important in the future for distinguishing among different kinds of models and testing particular schemes.

There are four indications of neutrino mass that have guided recent attempts to build models: (1) the solar neutrino problem, (2) the atmospheric neutrino anomaly, (3) the LSND experiment, and (4) dark matter. There are many excellent reviews of the evidence for neutrino mass.¹

(1) The three most promising solutions to the solar neutrino problem are based on neutrino mass. These are the small-angle MSW solution (SMA), the large-angle MSW solution (LMA), and the vacuum oscillation solution (VO). All these solutions involve ν_e oscillating into some other type of neutrino — in the models we shall consider, predominantly ν_μ . In the SMA solution the mixing angle and mass-squared splitting between ν_e and the neutrino into which it oscillates are roughly $\sin^2 2\theta \sim 5.5 \times 10^{-3}$ and $\delta m^2 \sim 5.1 \times 10^{-6} eV^2$. For the LMA solution one has $\sin^2 2\theta \sim 0.79$, and $\delta m^2 \sim 3.6 \times 10^{-5} eV^2$. (The numbers are best-fit values from a recent analysis.²) And for the VO solution $\sin^2 2\theta \sim 0.93$, and $\delta m^2 \sim 4.4 \times 10^{-10} eV^2$. (Again, these are best-fit values from a recent analysis.³)

(2) The atmospheric neutrino anomaly strongly implies that ν_μ is oscillating with nearly maximal angle into either ν_τ or a sterile neutrino, with the data preferring the former possibility.⁴ One has $\sin^2 2\theta \sim 1.0$, and $\delta m^2 \sim 3 \times 10^{-3} eV^2$.

(3) The LSND result, which would indicate a mixing between ν_e and ν_μ

with $\delta m^2 \sim 0.3 - 1 \text{eV}^2$ is regarded with more skepticism for two reasons. The experimental reason is that KARMEN has failed to corroborate the discovery, although it is true that KARMEN has not excluded the whole LSND region. The theoretical reason is that to account for the LSND result and also for both the solar and atmospheric anomalies by neutrino oscillations would require three quite different mass-squared splittings, and that can only be achieved with *four* species of neutrino. This significantly complicates the problem of model-building. In particular, it is regarded as not very natural, in general, to have a fourth sterile neutrino that is extremely light compared to the weak scale. (There are some theoretical frameworks that can give light sterile particles, but they tend to give many of them, not just one.) For these reasons, we assume that the LSND results do not need to be explained by neutrino oscillations, and the classification we present includes only three-neutrino models.

(4) The fourth possible indication of neutrino mass is the existence of dark matter. If a significant amount of this dark matter is in neutrino mass, it would imply a neutrino mass of order several eVs. In order then to achieve the small mass splittings needed to explain the solar and atmospheric anomalies one would have to assume that ν_e , ν_μ and ν_τ were nearly degenerate. We shall not focus on such models in our classification, which is primarily devoted to models with “hierarchical” neutrino masses. However, in most models with nearly degenerate masses, the neutrino mass matrix consists of a dominant piece proportional to the identity matrix and a much smaller hierarchical piece. Since the oscillations are caused by the small hierarchical piece, such models can be classified together with hierarchical models.

In sum, the models we shall classify are those which assume (a) three flavors of neutrino that oscillate (ν_e , ν_μ , and ν_τ), (b) the atmospheric anomaly explained by ν_μ - ν_τ oscillations with nearly maximal angle, and (c) the solar anomalies explained by ν_e oscillating primarily with ν_μ with either small angle (SMA) or large angle (LMA, VO).

There are several major divisions among models. One is between models in which the neutrino masses arise through the see-saw mechanism,⁵ and those in which the neutrino masses are generated directly at low energy. In see-saw models, there are both left- and right-handed neutrinos. Consequently, there are five fermion mass matrices to explain: the four Dirac mass matrices, U , D , L , and N of the up quarks, down quarks, charged leptons, and neutrinos, respectively, and the Majorana mass matrix M_R of

the right-handed neutrinos. The four Dirac mass matrices are all roughly of the weak scale, while M_R is generally much larger than the weak scale. After integrating out the superheavy right-handed neutrinos, the mass matrix of the left-handed neutrinos is given by $M_\nu = -N^T M_R^{-1} N$. In conventional see-saw models there are three right-handed neutrinos, one for each of the three families of quarks and leptons. And, typically, in such conventional see-saw models there is a close relationship between the 3×3 Dirac mass matrix N of the neutrinos and the other 3×3 Dirac mass matrices L , U , and D . Usually the four Dirac matrices are related to each other by grand unification and/or flavor symmetries. That means that in conventional see-saw models neutrino masses and mixings are just one aspect of the larger problem of quark and lepton masses, and are likely to shed great light on that problem, and perhaps even be the key to solving it. However, in most see-saw models the Majorana matrix M_R is either not related or is tenuously related to the Dirac mass matrices of the quarks and leptons. The freedom in M_R is the major obstacle to making precise predictions of neutrino masses and mixings in most see-saw schemes.

There are also what we shall refer to as unconventional see-saw models in which the fermions that play the role of the heavy right-handed partners of the neutrinos are not part of the ordinary three-family structure but are some other neutral fields. There need not be three of them, and the Dirac mass matrix of the neutrinos therefore need not be 3×3 nor need it have any particular connection to the other Dirac mass matrices L , U , and D . Such unconventional see-saw models we classify together with non-see-saw models.

In non-see-saw schemes, there are no right-handed neutrinos. Consequently, there are only four mass matrices to consider, the Dirac mass matrices of the quarks and charged leptons, U , D , and L , and the Majorana mass matrix of the light left-handed neutrinos M_ν . Typically in such schemes M_ν has nothing directly to do with the matrices U , D , and L , but is generated at low-energy by completely different physics.

The three most popular possibilities in recent models for generating M_ν at low energy in a non-see-saw way are (a) triplet Higgs, (b) variants of the Zee model,⁶ and (c) R-parity violating terms in low-energy supersymmetry. (a) In triplet-Higgs models, M_ν arises from a renormalizable term of the form $\lambda_{ij} \nu_i \nu_j H_T^0$, where H_T is a Higgs field in the $(1, 3, +1)$ representation of $SU(3) \times SU(2) \times U(1)$. (b) In the Zee model, the Standard Model is supplemented with a scalar, h , in the $(1, 1, +1)$ representation and having

weak-scale mass. This field can couple to the lepton doublets L_i as $L_i L_j h$ and to the Higgs doublets ϕ_a (if there is more than one) as $\phi_a \phi_b h$. Clearly it is not possible to assign a lepton number to h in such a way as to conserve it in both these terms. The resulting lepton-number violation allows one-loop diagrams that generate a Majorana mass for the left-handed neutrinos. (c) In supersymmetry the presence of such R-parity-violating terms in the superpotential as $L_i L_j E_k^c$ and $Q_i D_j^c L_k$, causes lepton-number violation, and allows one-loop diagrams that give neutrino masses. Neutrino mass can also arise at tree level from R-parity-violating terms of the form $H_u L_i$, which mix neutrino and Higgs superfields and lead to sneutrino vacuum expectation values.

It is clear that in all of these schemes the couplings that give rise to neutrino masses have little to do with the physics that gives mass to the other quarks and leptons. While this allows more freedom to the neutrino masses, it would from one point of view be very disappointing, as it would mean that the observation of neutrino oscillations is almost irrelevant to the great question of the origin of quark and charged lepton masses.

It should also be mentioned that some models derive the neutrino mass matrix M_ν directly from non-renormalizable terms of the form $\nu_i \nu_j H_u H_u / M$ without specifying where these terms come from. While such terms do arise in the conventional see-saw mechanism they can also arise in other ways. Models in which these operators do not arise from a see-saw or where their origin is left unspecified we classify as non-see-saw models.

Another major division among models has to do with the kinds of symmetries that constrain the forms of mass matrices and that, in some models, relate different mass matrices to each other. There are two main approaches: (a) grand unification, and (b) flavor symmetry. Many models use both.

(a) The simplest grand unified group is $SU(5)$. In minimal $SU(5)$ there is one relation among the Dirac mass matrices, namely $D = L^T$, coming from the fact that the left-handed charged leptons are unified with the right-handed down quarks in a $\bar{\mathbf{5}}$, while the right-handed charged leptons and left-handed down quarks are unified in a $\mathbf{10}$. In $SU(5)$ there do not have to be right-handed neutrinos, though they may be introduced. In $SO(10)$, which in several ways is a very attractive group for unification, the minimal model gives the relations $N = U \propto D = L$. In realistic models these relations are modified in various ways, for example by the appearance of Clebsch coefficients in certain entries of some of the mass matrices. It is clear that

unified symmetries are so powerful that very predictive models are possible. Most of the published models which give sharp predictions for masses and mixings are unified models.

(b) Flavor symmetries can be either abelian or non-abelian. Non-abelian symmetries are useful for obtaining the equality of certain elements of the mass matrix, as in models where the neutrino masses are nearly degenerate, and in the so-called “flavor democracy” schemes, which will be discussed later. Abelian symmetries are useful for explaining hierarchical mass matrices through the so-called Froggatt-Nielson mechanism.⁷ The idea is simply that different elements of the mass matrices arise at different orders in flavor symmetry breaking. In particular, different fermion multiplets can differ in charge under a $U(1)$ flavor symmetry that is spontaneously broken by some “flavon” expectation value (or values), $\langle f_i \rangle$. Thus, different elements of the fermion mass matrices would be suppressed by different powers of $\langle f_i \rangle / M \equiv \epsilon_i \ll 1$, where M is the scale of flavor physics. This kind of scheme can explain small mass ratios and mixings in the sense of predicting them to arise at certain orders in the small quantities ϵ_i . A drawback of such models compared to many grand unified models is that actual numerical predictions, as opposed to order of magnitude estimates, are not possible. On the other hand, models based on flavor symmetry involve less of a theoretical superstructure built on top of the Standard Model than do unified models, and could therefore be considered more economical in a certain sense. Unified models put more in but get more out than do abelian-flavor-symmetry models.

The most significant new fact about neutrino mixing is the largeness of the mixing between ν_μ and ν_τ . This comes as somewhat of a surprise from the point of view of both grand unification and flavor symmetry approaches. Since grand unification relates leptons to quarks, one might expect lepton mixing angles to be small like those of the quarks. In particular, the mixing between the second and third family of quarks is given by V_{cb} , which is known to be 0.04. That is to be compared to the nearly maximal mixing of the second and third families of leptons: $U_{\mu 3} \cong 1/\sqrt{2} \cong 0.7$. It is true that even in the early 1980’s some grand unified models predicted large neutrino mixing angles. (Especially noteworthy is the remarkably prophetic 1982 paper of Harvey, Ramond, and Reiss,⁸ which explicitly predicted and emphasized that there should be large $\nu_\mu - \nu_\tau$ mixing. However, in those days the top mass was expected to be light, and that paper assumed it to be 25 GeV. That gave V_{cb} to be about 0.22. The corresponding lepton mixing was further boosted

by a Clebsch of 3. With the actual value of m_t that we now know, the model of Ref. 8 would predict $U_{\mu 3}$ to be only 0.12). What makes the largeness of $U_{\mu 3}$ a puzzle in the present situation is the fact that we now know that both V_{cb} and m_c/m_t are exceedingly small.

The same puzzle exists in the context of flavor symmetry. The fact that the quark mixing angles are small suggests that there is a family symmetry that is only weakly broken, while the large mixings of some of the neutrinos would suggest that family symmetries are badly broken.

The first point of interest, therefore, in looking at any model of neutrino mixing is how it explains the large mixing of ν_μ and ν_τ . This will be the feature that we will use to organize the classification of models.

2 Classification of three-neutrino models

Virtually all three-neutrino models published in the last few years fit somewhere in the simple classification now to be described. In fact, almost all of them are cited below. The main divisions of this classification are based on how the large $\nu_\mu - \nu_\tau$ mixing arises. This mixing is described by the element $U_{\mu 3} \equiv \sin \theta_{23}$ of the so-called MNS matrix⁹ (analogous to the CKM matrix for the quarks).

The mixing angles of the neutrinos are the mismatch between the eigenstates of the neutrinos and those of the charged leptons, or in other words between the mass matrices L and M_ν . Thus, there are two obvious ways of obtaining large θ_{23} : either M_ν has large off-diagonal elements while L is nearly diagonal, or L has large off-diagonal elements and M_ν is nearly diagonal. Of course this distinction only makes sense in some preferred basis. But in almost every model there is some preferred basis given by the underlying symmetries of that model. This distinction gives the first major division in the classification, between models of what we shall call class I and class II. (It is also possible that the large mixing is due almost equally to large off-diagonal elements in L and M_ν , but this possibility seems to be realized in very few published models. We will put them into class II.)

If the large θ_{23} is due to M_ν (class I), then it becomes important whether M_ν arises from a non-see-saw mechanism or the see-saw mechanism. We therefore distinguish these cases as class I(1) and class I(2) respectively. In the see-saw models, M_ν is given by $-N^T M_R^{-1} N$, so a further subdivision is

possible: models in which the large mixing comes from large off-diagonal elements in M_R we call I(2A); models in which the large mixing comes from large off-diagonal elements in N we call I(2B); and models in which neither M_R nor N have large off-diagonal elements but $M_\nu = -N^T M_R^{-1} N$ nevertheless does we call I(2C).

The other main class of models, where θ_{23} is due to large off-diagonal elements in L the mass matrix of the *charged* leptons, we called class II. The question in these models is why, given that L has large off-diagonal elements, there are not also large off-diagonal elements in the Dirac mass matrices of the other charged fermions, especially D (which is typically closely related to L), causing large CKM mixing of the quarks. In the literature there seem to be two ways of answering this question. One way involves the CKM angles being small due to a cancellation between large angles that are nearly equal in the up and down quark sectors. We call this class II(1). The main examples of this idea are the so-called “flavor democracy models”. The other idea is that the matrices L and D^T (related by unified or flavor symmetry) are “lopsided” in such a way that the large off-diagonal elements only affect the mixing of fermions of one handedness: left-handed for the leptons, making $U_{\mu 3}$ large, and right-handed for the quarks, leaving V_{cb} small. We call this approach class II(2).

Schematically, one then has

- I* Large mixing from M_ν
 - (1) Non see saw
 - (2) See saw
 - A. Large mixing from M_R
 - B. Large mixing from N
 - C. Large mixing from $-N^T M_R^{-1} N$
 - II* Large mixing from L
 - (1) CKM small by cancellation
 - (2) lopsided L .
- (1)

Now let us examine the different categories in more detail, giving examples from the literature.

I(1) Large mixing from M_ν , non-see-saw.

This kind of model gives a natural explanation of the discrepancy between the largeness of $U_{\mu 3} = \sin \theta_{23}$ and the smallness of V_{cb} . V_{cb} comes from Dirac

mass matrices, which are all presumably nearly diagonal like L , whereas $U_{\mu 3}$ comes from the matrix M_ν ; and since in non-see-saw models M_ν comes from completely different physics than do the Dirac mass matrices it is not at all surprising that it has a very different form from the others, containing some large off-diagonal elements. While this basic idea is very simple and appealing, these models have the drawback that in non-see-saw models the form of M_ν , since it comes from new physics unrelated to the origin of the other mass matrices, is highly unconstrained. Thus, there are few definite predictions, in general, for masses and mixings in such schemes. However, in some schemes constraints can be put on the new physics responsible for M_ν .

As we saw, there are a variety of attractive ideas for generating a non-see-saw M_ν at low energy, and there are published models of neutrino mixing corresponding to all these ideas.¹⁰⁻²¹ M_ν comes from triplet Higgs in Refs. 10-12; from the Zee mechanism in Refs. 13-15; and from R-parity and lepton-number-violating terms in supersymmetry in Refs. 16 and 17.

In Ref. 18 a “democratic form” of M_ν is enforced by a family symmetry. (The democratic form is one in which all the elements of the matrix are equal or very nearly equal. In most schemes of “flavor democracy”, as we shall see later, it is the charged lepton mass matrix L that is assumed to have a democratic form and M_ν is assumed approximately diagonal, giving models of class II(1), But in Ref. 18 the opposite is assumed.) Several other models in class I(1) exist in the literature.^{19,20}

There is a basic question that has to be answered by any model of class I(1), namely why the mass splitting seen in solar neutrino oscillations (δm_{12}^2) is much smaller than that seen in atmospheric oscillations (δm_{23}^2). If all the elements of M_ν were of the same order, then indeed large mixing angles would be typical, as desired to explain the atmospheric neutrino oscillations, but the neutrino mass ratios would then also be typically of order unity, and one would expect $\delta m_{12}^2 \sim \delta m_{23}^2$. Conversely, if there is a small parameter in M_ν that accounts for the ratios of mass splittings, then the question arises why the mixing angles are not also controlled by that small parameter.

A satisfactory answer to these questions requires that M_ν have a special form. Three satisfactory forms are possible, as has been pointed out in several analyses.²¹ We shall consider them in turn.

(a) In the literature one finds that the majority of models of class I(1) (and, as we shall see later, many models of other classes too) assume the following form for M_ν :

$$M_\nu = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{12} & s^2 M + O(\delta)M & scM + O(\delta)M \\ m_{13} & scM + O(\delta)M & c^2 M + O(\delta)M \end{pmatrix}, \quad (2)$$

where $s \equiv \sin \theta$, $c \equiv \cos \theta$, $\theta \sim \pi/4$, $\delta \ll 1$, and $m_{ij} \ll M$. By a rotation in the 2-3 plane by an angle close to θ , the 2-3 block will be diagonalized and the matrix will take the form

$$M'_\nu \cong \begin{pmatrix} m_{11} & cm_{12} - sm_{13} & cm_{13} + sm_{12} \\ cm_{12} - sm_{13} & O(\delta)M & 0 \\ cm_{13} + sm_{12} & 0 & M \end{pmatrix}. \quad (3)$$

It is clear that for $m_{ij} \lesssim \delta M$ there is a hierarchy of mass eigenvalues, and that $\delta m_{23}^2 \cong M^2$ and $\delta m_{12}^2 = O(\delta^2)M^2$. On the other hand the atmospheric neutrino angle θ_{23} , which is approximately given by θ , is of order one. The value of the solar angle depends on the size of m_{ij} . In particular $\theta_{12} \sim (cm_{12} - sm_{13})/(\delta M)$. Consequently, either small angle or large angle solutions of the solar neutrino problem can be naturally obtained.

One sees that in order to get the hierarchy among neutrino mass splittings and at the same time a large atmospheric angle one has assumed a form for M_ν in Eq. (2) that has a special relationship among the 22, 23, 32, and 33 elements. If such a relationship existed simply accidentally, then the model would be “fine-tuned” to some extent. Specifically, the 2-3 block of M_ν would have a determinant that was of $O(\delta)$ times its “natural” value.

A number of the models in the literature that are of class I(1) are indeed fine-tuned in this way. However, two ways of achieving the special form in Eq. (2) in a technically natural way in models of class I(1) have been proposed in the literature: (i) factorization, and (ii) permutation symmetry.

(i) The idea of factorization is that the neutrino mass matrix arises from one-loop contributions that are dominated by a single diagram, giving $(M_\nu)_{ij} \cong \lambda_i \lambda_j M$, where λ_i is the coupling of ν_i to the particles in the loop. If $\lambda_2 \sim \lambda_3$ then the form in Eq. (2) results. A good example of this kind of model is Ref. 16, where λ_i is an R-parity-violating and lepton-number-violating coupling of ν_i to a quark-squark (or lepton-slepton) pair in supersymmetry.

A factorized form of M_ν can also arise at tree-level by a non-standard see-saw mechanism in which λ_i is a Dirac coupling of ν_i to a *single* heavy

Majorana fermion that is integrated out. This is the basic idea in the papers in Ref. 20. (A special case of this is supersymmetric models with R-parity-violating terms that mix neutrinos with other neutralinos. In these the neutralinos play the role of the heavy fermions in the see-saw, and factorized forms of M_ν can result.)

(ii) The other idea for achieving the form in Eq. (2) is permutation symmetry. The basic idea is to use non-abelian symmetry to relate different elements of M_ν . Generally the relationship will be one of equality, thus giving maximal mixing angles. A good example is the model of Ref. 10, in which an $S_2 \times S_2$ permutation symmetry among four left-handed neutrinos is used to obtain the form

$$M_\nu = \begin{pmatrix} A & B & C & D \\ B & A & D & C \\ C & D & A & B \\ D & C & B & A \end{pmatrix}. \quad (4)$$

Then by assuming that the linear combination of $(\nu_1 - \nu_2)/\sqrt{2}$ acquires a superlarge Majorana mass, the residual three light species of neutrino end up with a mass matrix

$$M'_\nu = \begin{pmatrix} A+B & F & F \\ F & A & B \\ F & B & A \end{pmatrix} = (A+B)I + \begin{pmatrix} 0 & F & F \\ F & -B & B \\ F & B & -B \end{pmatrix}, \quad (5)$$

which in effect has the form in Eq. (2), since the part proportional to the identity does not contribute to oscillations. From this one sees that it is possible to get the form in Eq. (2) in a technically natural way using flavor symmetries. Again, either small or large solar angle can arise depending on the magnitude of F/B .

(b) Another form for M_ν that is satisfactory is

$$M_\nu = \begin{pmatrix} m_{11} & cM & sM \\ cM & m_{22} & m_{23} \\ sM & m_{23} & m_{33} \end{pmatrix}, \quad (6)$$

where $m_{ij} \ll M$, and as before $s \equiv \sin \theta$ and $c \equiv \cos \theta$, with $\theta \sim \pi/4$. By a rotation in the 2-3 plane by angle θ one brings this to the form

$$M'_\nu = \begin{pmatrix} m_{11} & M & 0 \\ M & m'_{22} & m'_{23} \\ 0 & m'_{23} & m'_{33} \end{pmatrix}. \quad (7)$$

This pseudo-Dirac form in the 1-2 block shows that ν_1 and ν_2 will be maximally mixed with nearly degenerate masses approximately equal to M , while the third neutrino will have smaller mass. Thus $\delta m_{23}^2 \cong M^2$ and $\delta m_{12}^2 \sim m_{ij}M$. Such a form always gives bimaximal mixing, i.e. large mixing angle for both atmospheric neutrinos and solar neutrinos, in contrast to Eq. (2) which can give either large or small angle solutions for the solar neutrino problem.

The form in Eq. (6) can easily be achieved using various family symmetries.^{11,12,15} A particularly interesting possibility^{12,15} is that the symmetry in question is $L_e - L_\mu - L_\tau$, which if exact would allow only the large elements of order M in Eq. (6).

An interesting and instructive model in which M_ν is of the form given in Eq. (6) is found in Ref. 13. This model is based on the Zee mechanism, which gives a neutrino mass matrix M_ν that is symmetric but has vanishing diagonal elements. In particular it can give a matrix of the form

$$M_\nu \cong \begin{pmatrix} 0 & m/\sqrt{2} & -m/\sqrt{2} \\ m/\sqrt{2} & 0 & \Delta \\ -m/\sqrt{2} & \Delta & 0 \end{pmatrix}, \quad (8)$$

where $\Delta \ll m$. There is some mild fine-tuning in this model in the sense that in order for the 12 and 13 elements of M_ν to be nearly equal in magnitude (as must be so to have nearly maximal atmospheric angle) a relation among the couplings and masses of the Zee model must be satisfied that has no basis in symmetry.

(c) A third possible form for M_ν is

$$M_\nu = \begin{pmatrix} M' & m_{12} & m_{13} \\ m_{12} & m_{22} & M \\ m_{13} & M & m_{33} \end{pmatrix}, \quad (9)$$

where $m_{ij} \ll M$. In such a scheme, ν_μ and ν_τ are nearly maximally mixed and nearly degenerate, with $\delta m_{23}^2 \sim m_{ij}M \ll M^2$. Therefore, in order for the splitting δm_{12}^2 to be even smaller, it must be that $M' \cong M$ to great

accuracy. If this is not to be a fine-tuning of parameters, then it must be the consequence of some non-abelian flavor symmetry.

I(2A) See-saw M_ν , large mixing from M_R

In models of class of I(2), as in class I(1), the large atmospheric neutrino mixing angle comes from M_ν , which however is now assumed to arise from the conventional see-saw mechanism. M_ν therefore has the form $-N^T M_R^{-1} N$, where N is a 3×3 matrix typically related by symmetry to L , U , and D . In class I(2A), the large off-diagonal elements in M_ν are assumed to come from M_R , while the Dirac neutrino matrix N is assumed to be nearly diagonal and hierarchical like the other Dirac matrices L , U , and D . Many examples of models of class I(2A) exist in the literature.^{22–30} As with the models of class I(1), these models have the virtue of explaining in a natural way the difference between the lepton angle $U_{\mu 3}$ and the quark angle V_{cb} . The quark mixings all come from Dirac matrices, while the lepton mixings involve the Majorana matrix M_R , which it is quite reasonable to suppose might have a very different character, with large off-diagonal elements.

However, there is a general problem with models of this type, which not all the examples in the literature convincingly overcome. The problem is that if N has a hierarchical and nearly diagonal form, it tends to communicate this property to M_ν . For example, suppose we take $N = \text{diag}(\epsilon', \epsilon, 1)M$, with $1 \gg \epsilon \gg \epsilon'$. And suppose that the ij^{th} element of M_R^{-1} is called a_{ij} . Then the matrix M_ν will have the form

$$M_\nu \propto \begin{pmatrix} \epsilon'^2 a_{11} & \epsilon' \epsilon a_{12} & \epsilon' a_{13} \\ \epsilon' \epsilon a_{12} & \epsilon^2 a_{22} & \epsilon a_{23} \\ \epsilon' a_{13} & \epsilon a_{23} & a_{33} \end{pmatrix}. \quad (10)$$

If all the non-vanishing elements a_{ij} are of the same order of magnitude, then obviously M_ν is approximately diagonal and hierarchical. The contribution to the leptonic angles coming from M_ν would therefore typically be proportional to the small parameters ϵ and ϵ' . One way that a θ_{23} of $O(1)$ could arise is that the small parameter coming from N gets cancelled by a correspondingly large parameter from M_R^{-1} .²² The trouble is that to have such a relationship between the magnitudes of parameters in N and M_R is usually unnatural, since these matrices have very different origins. This problem has been pointed out by various authors.²³ We shall call it the Dirac-Majorana conspiracy problem.

This problem is avoided in models in which the hierarchies in N and M_R are controlled by the same family symmetries and the same small parameters. Example of such correlated hierarchies can be found in the papers of Ref. 24.

Another way of getting around the Dirac-Majorana conspiracy problem is to assume a special form for M_R . An apparently simple solution is to take the 2-3 block of M_R to be skew diagonal. For example, suppose

$$M_R \cong \begin{pmatrix} M' & 0 & 0 \\ 0 & 0 & M \\ 0 & M & 0 \end{pmatrix}, \quad N \cong \begin{pmatrix} \epsilon' & & \\ & \epsilon & \\ & & 1 \end{pmatrix}. \quad (11)$$

where $\epsilon' \ll \epsilon \ll 1$. Then the 2-3 block of $M_\nu = -N^T M_R^{-1} N$ is also approximately skew diagonal, and one has that ν_μ and ν_τ are nearly degenerate and maximally mixed, as needed to explain the atmospheric neutrino anomaly. A number of models in the literature exploit this idea.²⁵

Unfortunately, as most of the papers in Ref. 25 noted, this idea has a problem with solar neutrinos. The problem is that it is unnatural in such a scheme for the splitting δm_{12}^2 to be smaller than δm_{23}^2 . One has $m_2 \cong m_3 \cong \epsilon M$ and $m_1 \cong \epsilon'^2 M'$. Therefore, unless M' is tuned with great accuracy this scheme cannot give a satisfactory solution to the solar neutrino problem.

It is clear that if one seeks to avoid the Dirac-Majorana conspiracy problem and also to explain both solar and atmospheric neutrino oscillations, an even cleverer choice of the forms of N and M_R must be found. Several papers have found such forms.²⁶⁻²⁹ In the model of Ref. 27, for instance, the Dirac and Majorana matrices of the neutrinos have the forms

$$N = \begin{pmatrix} x^2 y & 0 & 0 \\ 0 & x & x \\ 0 & O(x^2) & 1 \end{pmatrix} m_D, \quad M_R = \begin{pmatrix} 0 & 0 & A \\ 0 & 1 & 0 \\ A & 0 & 0 \end{pmatrix} m_R, \quad (12)$$

where x and y are small parameters. If one computes $M_\nu = -N^T M_R^{-1} N$ one finds that

$$M_\nu = - \begin{pmatrix} 0 & O(x^4 y/A) & x^2 y/A \\ O(x^4 y/A) & x^2 & x^2 \\ x^2 y/A & x^2 & x^2 \end{pmatrix} m_D^2/m_R. \quad (13)$$

Observe that this gives a maximal mixing of the second and third families, without having to assume any special relationship between the small param-

eters in N (namely x , and y) and the parameter in M_R (namely A). This example is generalized in the papers of Ref. 28.

Note that the matrix in Eq. (13) is of the general form given in Eq. (2), but here it arises through the see-saw mechanism. An interesting point about the form of M_ν in Eq. (13) is that it gives bimaximal mixing. This is easily seen by doing a rotation of $\pi/4$ in the 2-3 plane, bringing the matrix to the form

$$M'_\nu = \begin{pmatrix} 0 & z & z' \\ z & 0 & 0 \\ z' & 0 & 2x^2 \end{pmatrix}. \quad (14)$$

In the 1-2 block this matrix has a Dirac form, giving nearly maximal mixing of ν_e .

Another interesting model that avoids the Dirac-Majorana conspiracy problem, but requires a mild fine-tuning to get the hierarchy among neutrino mass splittings, is given in Ref. 29. The Majorana and Dirac neutrino mass matrices in that model have the form

$$N = \begin{pmatrix} 0 & 0 & x \\ 0 & x & 0 \\ x & 0 & 1 \end{pmatrix} m, \quad M_R^{-1} = \begin{pmatrix} aM^{-1} & bM^{-1} & 0 \\ bM^{-1} & cM^{-1} & 0 \\ 0 & 0 & M'^{-1} \end{pmatrix}, \quad (15)$$

where $x \ll 1$, $M/M' \ll x^2$, and $a, b, c \sim 1$. This gives

$$M_\nu = \begin{pmatrix} \epsilon x^2 & 0 & \epsilon x \\ 0 & c & b \\ \epsilon x & b & a + \epsilon \end{pmatrix} (m^2 x^2 / M). \quad (16)$$

Here $\epsilon \equiv \frac{M}{M'} \frac{1}{x^2} \ll 1$. The atmospheric neutrino angle will be of order unity if a , b , and c are all of the same order, which requires no fine-tuning or Dirac-Majorana conspiracy. However, to make $\delta m_{12}^2 \ll \delta m_{23}^2$ requires that the condition $\sqrt{ac - b^2} \ll a, b, c$, which does not arise from any symmetry, be satisfied.

Models of class I(2A) can be constructed that predict either small or large values of the solar neutrino angle θ_{12} .

I(2B) See-saw M_ν , large mixing from N

We now turn to see-saw models in which the large atmospheric neutrino angle comes from large off-diagonal elements in the Dirac neutrino mass matrix N rather than in the Majorana matrix M_R .

At least at first glance, this seems to be a less natural approach. The point is that if the large θ_{23} is due to large off-diagonal elements in N , it might be expected that the other Dirac mass matrices, U , D , and L , would also have large off-diagonal elements, giving large CKM angles. The model in Ref. 31 only attempts to describe the lepton sector and so does not resolve this problem. In Ref. 32 it is assumed that N has large off-diagonal elements and L does not, but the difference in character of these matrices is not accounted for. In the interesting model of Ref. 33 the difference between N and the other Dirac matrices is accounted for by a fine-tuning. In that model all of the quark and lepton mass matrices are given (in terms of relatively few parameters) by linear combinations of certain matrices that are hierarchical and nearly diagonal. In order that N have off-diagonal elements that are comparable to its diagonal elements, an accidental cancellation must occur that suppresses the diagonal elements.

There are ways to construct models of class I(2B) in which the difference between N and the other Dirac matrices is explained without fine-tuning.³⁴ However, experience seems to show that this approach is harder to make work than the others.

I(2C) See-saw M_ν , large mixing from $-N^T M_R^{-1} N$

In order for the see-saw mass matrix $M_\nu = -N^T M_R^{-1} N$ to have large off-diagonal elements it is not necessary that either M_R or N have large off-diagonal elements, as emphasized in Ref. 35. Following Ref. 35, consider the matrices

$$N \sim \begin{pmatrix} \epsilon' & \epsilon' & \epsilon' \\ \epsilon' & \epsilon & \epsilon \\ \epsilon' & \epsilon & 1 \end{pmatrix} m, \quad M_R^{-1} = \begin{pmatrix} r_1 & 0 & 0 \\ 0 & r_2 & 0 \\ 0 & 0 & r_3 \end{pmatrix} M^{-1}, \quad (17)$$

where it is assumed that $r_2 \epsilon^2 \gg r_3, r_1 \epsilon^2$. Then to leading order in small quantities M_ν has the form

$$M_\nu \sim \begin{pmatrix} (\epsilon'/\epsilon)^2 & \epsilon'/\epsilon & \epsilon'/\epsilon \\ \epsilon'/\epsilon & 1 & 1 \\ \epsilon'/\epsilon & 1 & 1 \end{pmatrix} r_2 \epsilon^2 (m^2/M). \quad (18)$$

It is easy to understand what is happening. The fact that r_2 is much larger than r_1 and r_3 means that the right-handed neutrino of the second family is much lighter than the other two. Effectively, then, one right-handed neutrino dominates M_R^{-1} . As a consequence one obtains an approximately “factorized” form for M_ν , just as one did in the unconventional see-saw models considered in the papers of Ref. 20, in which a single right-handed fermion also dominated. Those unconventional see-saw models could also be considered as examples of class I(2C).

II(1) Large mixing from L , CKM small by cancellation

We now turn to those models in which the large value of θ_{23} comes predominantly from the charged lepton mass matrix L , with M_ν being nearly diagonal. The issue that arises in such models is whether the other Dirac mass matrices, especially D and U , also have large off-diagonal elements, and if so why this does not lead to large CKM angles for the quarks. Some published models do not deal with this question since they are only models of the lepton sector and do not attempt to describe the quarks at all.³⁶ However, while it may in non-see-saw models make sense to discuss M_ν apart from the other mass matrices, it would seem that under most reasonable assumptions the matrix L should have some relationship to U and D .

Why, then, are the CKM angles small? One possibility is that the CKM angles are small because of an almost exact cancellation between large angles needed to diagonalize U and D . That, in turn, would imply that U and D , even though highly non-diagonal, have nearly identical forms. This idea is realized in most so-called “flavor democracy” models.³⁷

A “flavor-democratic” mass matrix is one in which all the elements are equal:

$$M_{FD} \equiv \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} m_\ell \quad (19)$$

A Dirac mass matrix can have such a form as the result of separate S_3 permutation symmetries acting on the left-handed and right-handed fermions.

This form is of rank 1, and thus gives only one family a mass. Of course, in realistic models based on flavor democracy it is assumed that the mass matrices also get small corrections that come from the breaking of the permutation symmetries, which give rise to masses for the lighter families.

It is clear that the flavor democratic form is diagonalized by a unitary matrix with rotation angles that are large. In fact, the matrix is

$$U_{FD} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & -2/\sqrt{6} & 1/\sqrt{3} \end{pmatrix}. \quad (20)$$

The reason why the CKM angles are small in flavor democracy models is that both U and D are assumed to have approximately the democratic form. Thus, the large rotation angles nearly cancel between the up and down sectors. However, as first noted in Ref. 38, it is possible to have large neutrino mixing angles if M_ν is assumed to have not the democratic form but a nearly diagonal form. This difference in form is plausible, given that M_ν is a Majorana matrix rather than a Dirac matrix like the others. In this case, the angles required to diagonalize M_ν would be small, and the MNS matrix would come predominantly from diagonalizing L .

In Ref. 38, it is assumed that M_ν is diagonal and hierarchical in form, and the elements of M_ν are assumed to arise entirely from the breaking of the permutation symmetries of the model. The model of Ref. 38 therefore has the neutrino masses being hierarchical. However, most published versions of the flavor democracy idea³⁹ assume that M_ν is approximately proportional to the identity matrix. The form $M_\nu \propto I$ is invariant under an S_3 permutation of the left-handed neutrinos. (However, it should be noted that $M_\nu \propto I$ is not the most general form consistent with permutation symmetry, and so to make this form technically natural some further symmetries must be invoked.) Small deviations from the identity matrix would arise from terms that break the flavor symmetries of the model. In such versions, the three neutrino masses are nearly degenerate, but the splittings can be made hierarchical to accomodate the solar and atmospheric data.

An exact flavor-democratic form of L would leave two charged leptons degenerate and therefore one of the neutrino mixing angles undefined. And if M_ν were exactly proportional to the identity matrix, all three neutrinos would be degenerate and all three neutrino mixing angles would be undefined.

Exactly what angles are predicted for the neutrinos depends, therefore, on the form of the small contributions to the mass matrices that break the permutation symmetries. There are many possibilities. In some, the MNS matrix comes out to be very close to U_{FD}^\dagger . However, it is not surprising, given the degeneracies the the exact permutation-symmetric forms give, that the small permutation-symmetry-breaking contributions to the mass matrices can lead to additional large mixings, and to forms for the MNS matrix that depart significantly from U_{FD}^\dagger . It is typical in flavor democracy models for the angle θ_{12} to come out large, and in many cases it comes out to be close to $\pi/4$ as in the matrix U_{FD}^\dagger . However, it is possible for θ_{12} to be small. This can happen if the matrix M_ν is such that the neutrinos ν_1 and ν_2 form a pseudo-Dirac pair. Then the 1-2 angles from the diagonalization of both L and M_ν will be close to $\pi/4$ and their difference can be small.

The number of possible models, based on different ways to break permutation symmetry, is large. There exists an extensive and growing literature in this area. There are also many models based not on the pure flavor-democratic form in Eq. (19), but on forms in which all the elements of the mass matrix are assumed to be approximately equal in magnitude, but allowed to differ in complex phase. This is sometimes called the “Universal Strength for Yukawa couplings” approach or USY.⁴⁰

The idea of flavor democracy is an elegant one, especially in that it uses one basic idea to explain the largeness of the leptonic angles, the smallness of the quark angles, and the fact that one family is much heavier than the others. On the other hand, it is based on very special forms for the mass matrices which come from very specific symmetries. It is in this sense a narrower approach to the problem of fermion masses than some of the others we have discussed.

It would be interesting to know whether simple models of class II(1), in which the CKM angles are small by cancellations of large angles, can be constructed using ideas other than flavor democracy.

II(2) Large mixing from “lopsided” L

We now come to an idea for explaining the largeness of θ_{23} that has great flexibility, in the sense that it can be implemented in many different kinds of models: grand unified, models with abelian or non-abelian flavor symmetries, see-saw or non-see-saw neutrino masses, and so on. The basic idea of the “lopsided” L approach is that the charged-lepton and down-quark

mass matrices have the approximate forms

$$L \sim \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \epsilon \\ 0 & \sigma & 1 \end{pmatrix} m_D, \quad D \sim \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \sigma \\ 0 & \epsilon & 1 \end{pmatrix} m_D. \quad (21)$$

The “ \sim ” sign is used because in realistic models these σ and ϵ entries could have additional factors of order unity, such as from Clebsch coefficients. The fact that L is related closely in form to the *transpose* of D is a very natural feature from the point of view of $SU(5)$ or related symmetries, and is a crucial ingredient in this approach. The assumption is that $\epsilon \ll 1$, while $\sigma \sim 1$. In the case of the charged leptons ϵ controls the mixing of the second and third families of *right*-handed fermions (which is not observable at low energies), while σ controls the mixing of the second and third families of *left*-handed fermions, which contributes to θ_{23} and makes it large. For the quarks the reverse is the case because of the “ $SU(5)$ ” feature: the small $O(\epsilon)$ mixing is in the left-handed sector, accounting for the smallness of V_{cb} , while the large $O(\sigma)$ mixing is in the right-handed sector, where it cannot be observed and does no harm.

In this approach the three crucial elements are these: (a) Large mixing of neutrinos (in particular of ν_μ and ν_τ) caused by large off-diagonal elements in the *charged*-lepton mass matrix L ; (b) these off-diagonal elements appearing in a highly asymmetric or lopsided way; and (c) L being similar to the *transpose* of D by $SU(5)$ or a related symmetry.

What makes this approach so flexible is that the problem of obtaining a realistic pattern of neutrino masses is decoupled from the problem of getting large θ_{23} . The large θ_{23} arises from L while the neutrino mass spectrum arises from M_ν . Thus one is freed from having very special textures for M_ν as was the case in class I models. This is also true of the flavor democracy schemes; however, there the necessity of near cancellation between up and down quark angles forced a very particular kind of mass matrix texture and flavor symmetry. The lopsided mass matrices, by contrast, can be achieved in many ways, as can be seen from Refs. 41-47.

The first paper that has all three elements that define this approach seems to be Ref. 41, which proposed a very specific idea for generating the fermion mass hierarchies. The ideas of that paper were further explored in Ref. 42.

The lopsided L idea next is seen in three papers that appeared almost simultaneously.⁴³⁻⁴⁵ It is interesting that the same basic mechanism of lop-

sided L was arrived at independently by these three groups of authors from completely different starting points. In Ref. 43 the model is based on $E_7/SU(5) \times U(1)$, and the structure of the mass matrices is determined by the Froggatt-Nielson mechanism. In Ref. 44 the model is based on $SO(10)$, and does not use the Froggatt-Nielson approach. Rather, the constraints on the form of the mass matrices come from assuming a “minimal” set of Higgs for $SO(10)$ -breaking and choosing the smallest and simplest set of Yukawa operators that can give realistic mass matrices for the quarks and charged leptons. Though both Refs. 43 and 44 assume a unified symmetry larger than $SU(5)$, in both it is the $SU(5)$ subgroup that plays the critical role in relating L to D^T . The model of Ref. 45, like that of Ref. 43, uses the Froggatt-Nielson idea, but is not based on a grand unified group. Rather, the fact that L is related to D^T follows ultimately from the requirement of anomaly cancellation for the various $U(1)$ flavor symmetries of the model. However, it is well known that anomaly cancellation typically enforces charge assignments that can be embedded in unified groups. So that even though the model does not contain an explicit $SU(5)$, it could be said to be “ $SU(5)$ -like”.

In Ref. 46 are listed numerous papers that have used the lopsided L approach in the context of grand unified theories. A variety of symmetries — abelian, non-abelian continuous, and non-abelian discrete — are used in these models to constrain the forms of mass matrices. In Ref. 47 are papers that are not unified and do not discuss the quark mass matrices, so that the third element of the approach (L being related to D^T by a symmetry related to $SU(5)$) is not explicitly present.

As pointed out in Ref. 48, models based on lopsided L can give either large-angle or small-angle solutions to the solar neutrino problem.

A predictive model with lopsided L

We shall now briefly describe a particular model of class II(2). A remarkable fact about this model is that it was not constructed to explain neutrino phenomenology; rather it emerged from the attempt to find a realistic model of the masses of the charged leptons and quarks in the context of $SO(10)$. In fact, it is one of the most predictive models of quark and lepton masses that exists in the literature. The idea of the model was to take the Higgs sector of $SO(10)$ to be as minimal as possible, and then to find what this implied for the mass matrices of the quarks and charged leptons. In fact, in the first

paper proposing this model⁴⁹ no attention was paid to the neutrino mixings at all. Only subsequently was it noticed that the model actually predicts a large mixing of ν_μ with ν_τ and this led to a second paper, in which the implications for neutrino phenomenology were stressed.⁴⁴ The reason for the large mixing of ν_μ and ν_τ in this model is precisely the fact that the charged lepton mass matrix has a lopsided form.

The reason this lopsided form was built into the model of Refs. 44 and 49, was that it was necessary to account for certain well-known features of the mass spectrum of the quarks. In particular, the mass matrix entry that is denoted σ in Eq. (21) above plays three crucial roles in this model that have nothing to do with neutrino mixing. (1) It is required to get the Georgi-Jarlskog⁵⁰ factor of 3 between m_μ and m_s . (2) It explains the value of V_{cb} . (3) It explains why $m_c/m_t \ll m_s/m_b$. Remarkably, it turns out not only to perform these three tasks, but also gives mixing of order 1 between ν_μ and ν_τ . Not often are four birds killed with one stone.

In constructing the model, several considerations played a part. First, a “minimal” set of Higgs for $SO(10)$ was assumed. It has been shown⁵¹ that the smallest set of Higgs that will allow a realistic breaking of $SO(10)$ down to $SU(3) \times SU(2) \times U(1)$, with natural doublet-triplet splitting,⁵² consists of a single adjoint (**45**), two pairs of spinors (**16** + $\overline{\mathbf{16}}$), a pair of vectors (**10**), and some singlets. The adjoint, in order to give the doublet-triplet splitting, must have a VEV proportional to the $SO(10)$ generator $B - L$. This fact is an important constraint. Second, it was required that the qualitative features of the quark and lepton spectrum should not arise by artificial cancellations or numerical accidents. Third, it was required that the Georgi-Jarlskog factor arise in a simple and natural way. Fourth, it was required that the entries in the mass matrices should come from operators of low dimension that arise in simple ways from integrating out small representations of fermions.

Having imposed these conditions of economy and naturalness, a structure emerged that had just six effective Yukawa terms (just five if m_u is allowed to vanish). These gave the following mass matrices:

$$\begin{aligned}
U^0 &= \begin{pmatrix} \eta & 0 & 0 \\ 0 & 0 & \frac{1}{3}\epsilon \\ 0 & -\frac{1}{3}\epsilon & 1 \end{pmatrix} m_U, & D^0 &= \begin{pmatrix} 0 & \delta & \delta' \\ \delta & 0 & \sigma + \frac{1}{3}\epsilon \\ \delta' & -\frac{1}{3}\epsilon & 1 \end{pmatrix} m_D \\
N^0 &= \begin{pmatrix} \eta & 0 & 0 \\ 0 & 0 & -\epsilon \\ 0 & \epsilon & 1 \end{pmatrix} m_U, & L^0 &= \begin{pmatrix} 0 & \delta & \delta' \\ \delta & 0 & -\epsilon \\ \delta' & \sigma + \epsilon & 1 \end{pmatrix} m_D.
\end{aligned} \tag{22}$$

(The first papers^{49,44} gave only the structures of the second and third families, while this was extended to the first family in a subsequent paper.⁵³) Here $\sigma \cong 1.8$, $\epsilon \cong 0.14$, $\delta \cong |\delta'| \cong 0.008$, $\eta \cong 0.6 \times 10^{-5}$. The patterns that are evident in these matrices are due to the $SO(10)$ group-theoretical characteristics of the various Yukawa terms. Notice several facts about the crucial parameter σ that is responsible for the lopsidedness of L and D . First, if σ were not present, then instead of the Georgi-Jarlskog factor of 3, the ratio m_μ/m_s would be given by 9. (That is, the Clebsch of $\frac{1}{3}$ that appears in D due to the generator $B - L$ gets squared in computing m_s .) Since the large entry σ overpowers the small entries of order ϵ , the correct Georgi-Jarlskog factor emerges. Second, if σ were not present, U and D would be proportional, as far as the two heavier families are concerned, and V_{cb} would vanish. Third, by having $\sigma \sim 1$ one ends up with V_{cb} and m_s/m_b being of the same order ($O(\epsilon)$) as is indeed observed. And since σ does not appear in U (for group-theoretical reasons) the ratio m_c/m_t comes out much smaller, of $O(\epsilon^2)$, also as observed. In fact, with this structure, the mass of charm is predicted correctly to within the level of the uncertainties.

Thus, for several reasons *that have nothing to do with neutrinos* one is led naturally to exactly the lopsided form that is found to give an elegant explanation of the mixing seen in atmospheric neutrino data.

From the very small number of Yukawa terms, and from the fact that $SO(10)$ symmetry gives the normalizations of these terms, and not merely order of magnitude estimates for them, it is not surprising that many precise predictions result. In fact there are altogether nine predictions.⁵³ Some of these are post-dictions (including the highly non-trivial one for m_c). But several predictions will allow the model to be tested in the future, including predictions for V_{ub} , and the mixing angles U_{e2} U_{e3} .

3 Expectations for the parameter U_{e3}

All of the models that we have discussed aim to explain the atmospheric neutrino anomaly by saying that there is maximal mixing between ν_μ and ν_τ , i.e. that $U_{\mu 3} \cong 1/\sqrt{2}$, and they all aim to explain the solar neutrino problem either by the small-angle MSW solution, in which $U_{e2} \simeq 0.05$, or by one of the large-angle solutions (large-angle MSW or vacuum oscillation), in which $U_{e2} \simeq 1/\sqrt{2}$. In this section we examine the other mixing, which is described by U_{e3} . U_{e3} is independent of the other two mixings, and *a priori* could take values ranging from zero up to the present limit of about 0.2. However, what we find is that the great majority of viable models give one of four mixing patterns, which we label with the Greek letters α through δ . These patterns are indicated in the following table.

<i>Pattern</i>	U_{e2}	U_{e3}
α	$\sin \theta_{LA}$	$O(m_{\nu_\mu}/m_{\nu_\tau}), \quad O(\sqrt{m_e/m_\mu}) \sim 0.05$
α'	$\sin \theta_{LA}$	$\sqrt{m_e/m_\mu} \sin \theta_{atm}$
α''	$\sin \theta_{LA}$	$\frac{2}{\sqrt{6}} \sqrt{m_e/m_\mu}$
β	$\sin \theta_{LA}$	0
γ	$\sin \theta_{SA}$	$\sin \theta_{SA} \tan \theta_{atm}$
γ'	$\sqrt{m_e/m_\mu} \cos \theta_{atm}$	$\sqrt{m_e/m_\mu} \sin \theta_{atm}$
δ	$\sin \theta_{SA}$	$\ll \sin \theta_{SA} \tan \theta_{atm}$

In this table θ_{SA} and θ_{LA} stand for the value of θ_{12} in the small-angle and large-angle solutions of the solar neutrino problem, respectively. What one sees is that if the solar angle is maximal one expects either that U_{e3} will be of order 0.05 (pattern α) or that it will vanish (pattern β). In most models

that fit pattern α only the order of magnitude of U_{e3} is predicted. However, some models predict it sharply. A particularly interesting prediction that arises in certain types of models is that $|U_{e3}| = \sqrt{m_e/m_\mu} \sin \theta_{atm}$, which we distinguish with the name α' . A special case of this occurs in some flavor democracy models, where $\sin \theta_{atm} = 2/\sqrt{6}$, and we call this α'' .

If the solar angle is small, i.e. the small-angle MSW solution, one typically finds one of two results for U_{e3} : either it is given by the relation $U_{e3} \cong U_{e2} \tan \theta_{atm}$ (pattern γ) or it is small compared to that value (pattern δ). In certain models with pattern γ there is the further prediction for the solar angle that $U_{e2} \cong \sqrt{m_e/m_\mu} \cos \theta_{atm}$. We call this pattern γ' , and it leads to the same prediction for U_{e3} that one has in pattern α' .

We will first give some general preliminaries and then proceed to analyze different kinds of models, showing why the four patterns we have described are the ones that arise in the great majority of published models.

The lepton mixing matrix, or “MNS matrix” has the form

$$U_{MNS} = U_L^\dagger U_\nu, \quad (23)$$

where U_L is the unitary matrix that diagonalizes $L^\dagger L$, and U_ν is the unitary matrix that diagonalizes $M_\nu^\dagger M_\nu$. It is convenient to write U_L in the form

$$U_L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \bar{c}_{23} & \bar{s}_{23} \\ 0 & -\bar{s}_{23} & \bar{c}_{23} \end{pmatrix} \begin{pmatrix} \bar{c}_{13} & 0 & \bar{s}_{13} \\ 0 & 1 & 0 \\ -\bar{s}_{13} & 0 & \bar{c}_{13} \end{pmatrix} \begin{pmatrix} \bar{c}_{12} & \bar{s}_{12} & 0 \\ -\bar{s}_{12} & \bar{c}_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (24)$$

where $\bar{s}_{ij} \equiv \sin \bar{\theta}_{ij}$, and so on. One can write U_ν in a similar way, with the corresponding angles being denoted $\tilde{\theta}_{ij}$. (Henceforth, a bar over a quantity means that it comes from the charged lepton sector, while a tilde means that it comes from the neutrino sector.) Consequently, if we assume all quantities are real the MNS matrix can be written

$$U_{MNS} = \begin{pmatrix} \bar{c}_{13}\bar{c}_{12} & -\bar{s}_{12} & -\bar{s}_{13}\bar{c}_{12} \\ -\bar{c}_{13}\bar{s}_{12} & \bar{c}_{12} & -\bar{s}_{13}\bar{s}_{12} \\ \bar{s}_{13} & 0 & \bar{c}_{13} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} \tilde{c}_{13}\tilde{c}_{12} & \tilde{c}_{13}\tilde{s}_{12} & \tilde{s}_{13} \\ -\tilde{s}_{12} & \tilde{c}_{12} & 0 \\ -\tilde{s}_{13}\tilde{c}_{12} & -\tilde{s}_{13}\tilde{s}_{12} & \tilde{c}_{13} \end{pmatrix}, \quad (25)$$

where $s_{23} \equiv \sin(\tilde{\theta}_{23} - \bar{\theta}_{23})$. What makes these expressions useful is that for hierarchical mass matrices, and most of the mass matrices that we shall have

to deal with, the angles $\bar{\theta}_{ij}$ and $\tilde{\theta}_{ij}$ are given with sufficient accuracy very simply in terms of ratios of elements of the mass matrices. Equation (25) tells us that

$$\begin{aligned}
U_{e2} &= \bar{c}_{13}\bar{c}_{12}\tilde{c}_{13}\tilde{s}_{12} - \bar{s}_{12}c_{23}\tilde{c}_{12} + \bar{s}_{12}s_{23}\tilde{s}_{13}\tilde{s}_{12} + \bar{s}_{13}\bar{c}_{12}s_{23}\tilde{c}_{12} + \bar{s}_{13}\bar{c}_{12}c_{23}\tilde{s}_{13}\tilde{s}_{12} \\
U_{e3} &= -\bar{s}_{12}s_{23}\tilde{c}_{13} - \bar{s}_{13}\bar{c}_{12}c_{23}\tilde{c}_{13} + \bar{c}_{13}\bar{c}_{12}\tilde{s}_{13} \\
U_{\mu3} &= \bar{c}_{12}s_{23}\tilde{c}_{13} + \bar{c}_{13}\bar{s}_{12}\tilde{s}_{13} - \bar{s}_{13}\bar{s}_{12}c_{23}\tilde{c}_{13}.
\end{aligned} \tag{26}$$

As we shall now see, for the practically interesting cases these expressions can be greatly simplified due to the smallness of certain angles. First let us consider the form in Eq. (2). From Eq. (2) one sees immediately that $\tilde{s}_{23} \cong s$. From Eq. (3) one sees that $\tilde{s}_{13} \cong (cm_{13} + sm_{12})/m_3$ and $\tilde{s}_{12} \cong (cm_{12} - sm_{13})/m_2$, where m_2 and m_3 are the second and third eigenvalues of M_ν . ($m_3 \cong M$ and $m_2 = O(\delta)M$.) Since $s, c \sim 1$, one expects that $cm_{13} + sm_{12} \sim cm_{12} - sm_{13}$, unless there is fine-tuning. Consequently, one expects that $\tilde{s}_{13} \sim (m_2/m_3)\tilde{s}_{12} \ll \tilde{s}_{12}$.

The same is true in virtually all models for the charged lepton sector, i.e. $\bar{s}_{13} \sim (m_\mu/m_\tau)\bar{s}_{12} \ll \bar{s}_{12}$. It is also usually true in models (except the flavor democracy models) that if the 1-2 mixing is large it is due to the angle $\tilde{\theta}_{12}$ being large rather than the angle $\bar{\theta}_{12}$. In other words, \tilde{s}_{12} may be large or small depending on which solution to the solar neutrino problem is assumed, but \bar{s}_{12} is small in almost all models (except the flavor democracy ones), implying that \bar{s}_{13} is even smaller and in fact negligible. However, \tilde{s}_{13} can be significant if $\tilde{s}_{12} \sim 1$. These facts allow one to write

$$\begin{aligned}
U_{e2} &\cong \tilde{s}_{12} - \bar{s}_{12}c_{23}\tilde{c}_{12} \\
U_{e3} &\cong -\bar{s}_{12}s_{23} + \tilde{s}_{13} \\
U_{\mu3} &\cong s_{23} + \bar{s}_{12}\tilde{s}_{13}
\end{aligned} \tag{27}$$

Now let us turn to models of class I(1) that have M_ν of the form given in Eq. (2). There are two cases to consider: either large- or small-angle solution to the solar neutrino problem. If small-angle then one has that $U_{e2} \sim 0.05$, and therefore, barring accidental cancellations, $\bar{s}_{12}, \tilde{s}_{12} \lesssim 0.05$.

Thus $\tilde{s}_{13} \ll 0.05$ and the formulas can be simplified to $U_{e3} \cong -\bar{s}_{12}s_{23}$ and $U_{e2} \cong -\bar{s}_{12}c_{23} + \tilde{s}_{12}$. If the solar neutrino angle is predominantly from the charged lepton sector, i.e. $\bar{s}_{12} \gg \tilde{s}_{12}$, then one has the predictions that $U_{e2} \cong -\bar{s}_{12}c_{23}$ and $U_{e3} \cong -\bar{s}_{12}s_{23}$, and therefore $U_{e3} \cong U_{e2} \tan \theta_{23} \cong U_{e2} \tan \theta_{atm}$. In other words, we have the mixing pattern γ . It is known experimentally that $c_{23} \approx 0.7$ and that (for small-angle MSW) $U_{e2} \approx 0.05$, and so these relations imply that $|\bar{s}_{12}| \cong |U_{e2}/c_{23}| \approx 0.07$. It is quite interesting that this is numerically close to $\sqrt{m_e/m_\mu}$. The relation $\bar{s}_{12} \cong \sqrt{m_e/m_\mu}$ is what would be obtained in models where the 1-2 block of the charged-lepton mass matrix has the Weinberg-Wilczek-Zee-Fritzsch form.⁵⁴ In such models one can get the fairly sharp predictions for both U_{e2} and U_{e3} that we call pattern γ' . The very interesting point that $U_{e2} = \sqrt{m_e/m_\mu} \cos \theta_{23}$ can arise in a simple way and that it gives a good fit for the small-angle MSW solution was first emphasized in Ref. 29. One of the models that gives the pattern γ' predictions is the small-angle case of the model of Refs. 44 and 53.

The other possibility in the small solar angle case is that the solar angle comes predominantly from the neutrino sector, i.e. $\tilde{s}_{12} \gg \bar{s}_{12}$. Then it is apparent that one would have $U_{e3} \ll U_{e2}$, in other words what we called mixing pattern δ . Of course, one could have $\bar{s}_{12} \cong \tilde{s}_{12}$, but such a coincidence is not what one would typically expect.

Next let us consider models of class I(1) with M_ν of the form given in Eq. (2) but with large-angle solar solution. In that case, as noted, in virtually all published models the large solar angle comes from the neutrino sector. Thus $\tilde{s}_{12} \sim 1$ and $\bar{s}_{12} \ll 1$. One then expects, as seen above, that $\tilde{s}_{13} \sim (m_2/m_3)\tilde{s}_{12} \sim m_2/m_3$, which for hierarchical models is $m_{\nu_\mu}/m_{\nu_\tau} \cong (\delta m_{12}^2/\delta m_{23}^2)^{1/2}$. For large-angle MSW solution to the solar neutrino problem this gives $\tilde{s}_{13} \sim 0.05$. One typically finds in most models that $\bar{s}_{12} \sim \sqrt{m_e/m_\mu} \cong 0.07$. Thus the two terms in the expression $U_{e3} \cong -\bar{s}_{12}s_{23} + \tilde{s}_{13}$ are typically of the same order but not sharply predicted. Consequently, all one can say is that $U_{e3} = O(m_{\nu_\mu}/m_{\nu_\tau})$ or $\sqrt{m_e/m_\mu}$. In other words, one has what we called mixing pattern α . Similar results follow for the vacuum oscillation solution with hierarchical neutrino masses. In that case, however, δm_{12}^2 is much smaller, so that one has $\tilde{s}_{13} \sim 10^{-4}$, which is negligible. Therefore U_{e3} comes from the single term $-\bar{s}_{12}s_{23}$. In most models one has no sharp prediction for this, and therefore the mixing pattern is again α . However, in some models having the WWZF form for the

1-2 block of L it is predicted that $\bar{s}_{12} \cong \sqrt{m_e/m_\mu}$, in which case the mixing pattern is α' . (A good example of a model with pattern α' is the large-angle version of the model in Refs. 44 and 53. This version is discussed in Refs. 48 and 55.)

So far we have been considering models of class I(1) in which the matrix M_ν has the form given in Eq. (2). Now let us consider the form given in Eq. (6). This form only gives large-angle solutions to the solar neutrino problem. It is apparent by inspection of Eq. (6) that $\delta m_{23}^2 \cong M^2$, while $\delta m_{12}^2 \sim m_{ij}M$. (More precisely, it turns out that $\delta m_{12}^2 \cong 2(m_{11} + c^2 m_{22} + 2scm_{23} + s^2 m_{33})M$.) Thus typically $m_{ij}/M \sim \delta m_{12}^2/\delta m_{23}^2 \sim 10^{-3}$ or 10^{-7} for large-angle MSW and vacuum oscillation solutions respectively. It is straightforward to show that Eq. (6) gives $\tilde{s}_{13} \cong (-scm_{22} + (c^2 - s^2)m_{23} + scm_{33})/M$. Consequently, unless there is some artificial tuning of the m_{ij} one can conclude that $\tilde{s}_{13} \lesssim 10^{-3}$ and hence negligible. Therefore, $U_{e3} \cong -\bar{s}_{12}s_{23}$. Generally, this means mixing pattern α , but where \bar{s}_{12} is predicted to be $\sqrt{m_e/m_\mu}$ one has pattern α' .

This brings us to models of class I(2). As can be seen from Eqs. (13), (15), and (18) most models of this class that do not involve fine-tuning seem to yield the form for M_ν given in Eq. (2). These models give the same results for U_{e3} as do class I(1) models that have the form in Eq. (2). The same is also effectively true for most models of class II(2), namely the models with lopsided L . It is true that in class II(2) models the large 2-3 mixing comes from the charged lepton sector rather than from M_ν . However, as can be seen from Eq. (25) it does not much matter in computing the MNS matrix where the 2-3 mixing originates. In class II(2) models, if the 2-3 block of the charged-lepton mass matrix L is diagonalized, the matrix M_ν generally goes over to the form in Eq. (2). (This will be the case if the neutrino masses have the hierarchy $m_3 \gg m_2 \gg m_1$, as typically is the case in class II(2).)

Let us consider, finally, the models of class II(1). Almost all published models of this class are of the “flavor democracy” type, as we have seen. Up to now we have analyzed predictions for U_{e3} using the forms given in Eqs. (24) and (25). However, these forms are convenient when the mass matrices have a hierarchy among their elements, which is not the case for the flavor democratic form, Eq. (19). Therefore we shall analyze the flavor democracy models in a different way.

In flavor democracy models, it is assumed that the lepton mass matrices have the following forms

$$\begin{aligned}
L &= M_{FD} + \Delta L, \\
M_\nu &= m_\nu I + \Delta M_\nu,
\end{aligned} \tag{28}$$

where M_{FD} is the form in Eq. (19), I is the identity matrix, and ΔL and ΔM_ν are small corrections that break the flavor permutation symmetries of the model (generally $S_3 \times S_3$). In Ref. 38 the parameter m_ν vanishes and $M_\nu = \Delta M_\nu$ has a hierarchical form, thus giving $m_3 \ll m_2 \ll m_1$ for the three neutrino masses. But the more usual assumption is that $m_\nu \neq 0$, giving $m_3 \cong m_2 \cong m_1$. However, there is still assumed to be a hierarchy in ΔM_ν so as to get $\delta m_{12}^2 \ll \delta m_{23}^2$.

The first step in diagonalizing L is to transform it by the orthogonal matrix U_{FD} given in Eq. (20).

$$\begin{aligned}
L' \equiv U_{FD}^\dagger L U_{FD} &= U_{FD}^\dagger (M_{FD} + \Delta L) U_{FD} \\
&= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix} m_\ell + \Delta L'.
\end{aligned} \tag{29}$$

Define

$$(\Delta L)_{ij} \equiv \delta_{ij} m_\ell, \quad (\Delta L')_{ij} \equiv \delta'_{ij} m_\ell, \tag{30}$$

Then the δ'_{ij} are given by

$$\begin{aligned}
\delta'_{11} &= \frac{1}{2}(\delta_{11} + \delta_{22} - \delta_{12} - \delta_{21}), \\
\delta'_{22} &= \frac{1}{6}(\delta_{11} + \delta_{22} + \delta_{12} + \delta_{21}) \\
&\quad - \frac{1}{3}(\delta_{13} + \delta_{31} + \delta_{23} + \delta_{32}) + \frac{2}{3}\delta_{33}, \\
\delta'_{33} &= \frac{1}{3}\sum_{ij}\delta_{ij}, \\
\delta'_{12} &= \frac{1}{2\sqrt{3}}(\delta_{11} - \delta_{22} + \delta_{12} - \delta_{21} - 2\delta_{13} + 2\delta_{23}), \\
\delta'_{13} &= \frac{1}{\sqrt{6}}(\delta_{11} - \delta_{22} + \delta_{12} - \delta_{21} + \delta_{13} - \delta_{23}), \\
\delta'_{23} &= \frac{1}{3\sqrt{2}}\sum_i(\delta_{1i} + \delta_{2i} - 2\delta_{3i}).
\end{aligned} \tag{31}$$

The next step in the diagonalization is to rotate away the 13, 31, 23, and 32 elements of L' as follows

$$\begin{aligned}
L'' &= \begin{pmatrix} 1 & 0 & -\delta'_{13}/3 \\ 0 & 1 & -\delta'_{23}/3 \\ \delta'_{13}/3 & \delta'_{23}/3 & 1 \end{pmatrix} L' \begin{pmatrix} 1 & 0 & \delta'_{31}/3 \\ 0 & 1 & \delta'_{32}/3 \\ -\delta'_{31}/3 & -\delta'_{32}/3 & 1 \end{pmatrix} \\
&\cong \begin{pmatrix} \delta'_{11} & \delta'_{12} & 0 \\ \delta'_{21} & \delta'_{22} & 0 \\ 0 & 0 & 3 \end{pmatrix} m_\ell.
\end{aligned} \tag{32}$$

Finally the 1-2 block of L'' is diagonalized

$$L_{diag} = \begin{pmatrix} \cos \theta' & -\sin \theta' & 0 \\ \sin \theta' & \cos \theta' & 0 \\ 0 & 0 & 1 \end{pmatrix} L'' \begin{pmatrix} \cos \theta_\ell & \sin \theta_\ell & 0 \\ -\sin \theta_\ell & \cos \theta_\ell & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{33}$$

where

$$\tan 2\theta_\ell = \frac{2(\delta'_{11}\delta'_{12} + \delta'_{21}\delta'_{22})}{(\delta'^2_{22} + \delta'^2_{12} - \delta'^2_{21} - \delta'^2_{11})}. \tag{34}$$

As emphasized in Ref. 56, there is no reason *a priori* for this angle to be small, a point to which we shall return presently.

Altogether, then, the matrix U_L that diagonalizes $L^\dagger L$ is given by

$$U_L = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} 1 & 0 & \delta'_{31}/3 \\ 0 & 1 & \delta'_{32}/3 \\ -\delta'_{31}/3 & -\delta'_{32}/3 & 1 \end{pmatrix} \begin{pmatrix} \cos \theta_\ell & \sin \theta_\ell & 0 \\ -\sin \theta_\ell & \cos \theta_\ell & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{35}$$

The usual assumption is that M_ν is nearly diagonal, so that $U_\nu \cong I$ and the MNS matrix is given by $U_{MNS} = U_L^\dagger U_\nu \cong U_L^\dagger$. From Eq. (35) one has then

$$\begin{aligned}
U_{\mu 3} &\cong -2 \cos \theta_\ell / \sqrt{6} + O(\delta), \\
U_{e 2} &\cong -\cos \theta_\ell / \sqrt{2} - \sin \theta_\ell / \sqrt{6} + O(\delta), \\
U_{e 3} &\cong 2 \sin \theta_\ell / \sqrt{6} - (\sin \theta_\ell \delta'_{32} - \cos \theta_\ell \delta'_{31}) / 3\sqrt{3}.
\end{aligned} \tag{36}$$

Since the angle θ_ℓ is very sensitive to the nine parameters δ_{ij} and has no reason *a priori* to be small, as is apparent from Eq. (34), it might seem

that the flavor democracy idea has no predictivity as far as the MNS matrix elements are concerned. However, *a posteriori* we do know that the CKM angles are small, and that strongly suggests that θ_ℓ is small. The point is that if a large angle θ_ℓ were required in the diagonalization of L , one would typically expect to find that large angles θ_u and θ_d were required in the diagonalization of U and D as well. Unless there were a conspiracy and $\theta_u \cong \theta_d$, large CKM angles would result. Under the assumption that ΔL has the same form (with different values of parameters) as ΔU and ΔD , one can conclude that $\theta_\ell \ll 1$.

There are many possible forms for ΔL that give vanishing θ_ℓ . If such a form is chosen, then one has

$$\begin{aligned} U_{\mu 3} &= -2/\sqrt{6} + O(\delta), \\ U_{e 2} &= -1/\sqrt{2} + O(\delta), \\ U_{e 3} &= O(\delta). \end{aligned} \tag{37}$$

The exact value is evidently dependent on the scheme of symmetry breaking. However, since the parameters δ_{ij} are involved in generating the interfamily hierarchy of charged lepton masses, one expects that U_{e3} will be closely related to small lepton mass ratios. In fact, this is the case, and typically one finds that $U_{e3} \sim \sqrt{m_e/m_\mu}$, in other words pattern α . In a popular scheme of symmetry breaking,^{57,58} for instance, $|U_{e3}| \cong \frac{2}{\sqrt{6}}\sqrt{m_e/m_\mu}$, which we have called pattern α'' . However, there are also schemes of symmetry breaking⁵⁷ where $U_{e3} = 0$, which we called pattern β .

In conclusion, we see that there are a few patterns of neutrino mixing that tend to arise in the great majority of published models. And although there is not a one-to-one correspondence between the type of model and the value of U_{e3} , it is clear that knowledge of U_{e3} will give great insight into the possible underlying mechanisms that are responsible for neutrino mixing.⁵⁹

References

1. J.W.F. Valle, hep-ph/9911224; S.M. Bilenky, Lectures at the 1999 European School of High Energy Physics, Casta Papiernicka, Slovakia,

- Aug. 22-Sept. 4, 1999 hep-ph/0001311.
2. M.C. Gonzalez-Garcia, P.C. de Holanda, C. Peña-Garay, and J.C.W. Valle, hep-ph/9906469
 3. V. Barger and K. Whisnant, hep-ph/9903262
 4. M.C. Gonzalez-Garcia, talk at International Workshop on Particles in Astrophysics and Cosmology: From Theory to Observation, Valencia, Spain, May 3-8, 1999.
 5. M. Gell-Mann, P. Ramond, and R. Slansky, in *Supergravity, Proc. Supergravity Workshop at Stony Brook*, ed. P. Van Nieuwenhuizen and D.Z. Freedman (North-Holland, Amsterdam (1979)); T. Yanagida, *Proc. Workshop on unified theory and the baryon number of the universe*, ed. O. Sawada and A. Sugamoto (KEK, 1979).
 6. A. Zee, *Phys. Lett.* **B93**, 389 (1980); *Phys. Lett.* **B161**, 141 (1985).
 7. C. Froggatt and H.B. Nielson, *Nucl. Phys.* **B147**, 277 (1979).
 8. J.A. Harvey, D.B. Reiss, and P. Ramond, *Nucl. Phys.* **B199**, 223 (1982).
 9. Z. Maki, M. Nakagawa, and S. Sakata, *Prog. Theor. Phys.* **28**, 870 (1962).
 10. R.N. Mohapatra and S. Nussinov, *Phys. Rev.* **D60**, 013002 (1999) (hep-ph/9809415).
 11. C.D. Froggatt, M. Gibson, and H.D. Nielson, *Phys. Lett.* **B446**, 256 (1999) (hep-ph/9811265).
 12. A.S. Joshipura, hep-ph/9808261; A.S. Joshipura and S.D. Rindani, hep-ph/9811252; R.N. Mohapatra, A. Perez-Lorenzana, C.A. deS. Pires, *Phys. Lett.* **B474**, 355 (2000) (hep-ph/9911395).
 13. C. Jarlskog, M. Matsuda, S. Skadhauge, and M. Tanimoto, *Phys. Lett.* **B449**, 240 (1999) (hep-ph/9812282).

14. E. Ma, *Phys. Lett.* **B442**, 238 (1998) (hep-ph/9807386); K. Cheung and O.C.W. Kong, hep-ph/9912238.
15. P. Frampton and S. Glashow, *Phys. Lett.* **B461**, 95 (1999) (hep-ph/9906375); A.S. Joshipura and S.D. Rindani, *Phys. Lett.* **B464**, 239 (1999) (hep-ph/9907390).
16. M. Drees, S. Pakvasa, X. Tata, T. terVeldhuis, *Phys. Rev.* **D57** 5335 (1998) (hep-ph/9712392).
17. E.J. Chun, S.K. Kang, C.W. Kim, and U.W. Lee, *Nucl. Phys.* **B544**, 89 (1999) (hep-ph/9907327); A.S. Joshipura and S.K. Vempati, *Phys. Rev.* **D60**, 095009 (1999) (hep-ph/9808232); B. Mukhopadhyaya, S. Roy, and F. Vissani, *Phys. Lett.* **B443**, 191 (1998) (hep-ph/9808265); O.C.W. Kong, hep-ph/9808304; K. Choi, E.J. Chun, and K. Hwang, *Phys. Rev.* **D60**, 031301 (1999) (hep-ph/9811363); D.E. Kaplan and A.E. Nelson, *JHEP* 0001:033 (2000) (hep-ph/9901254); A.S. Joshipura and S.K. Vempati, *Phys. Rev.* **D60**, 111303 (1999) (hep-ph/9903435); J.C. Romao, M.A. Diaz, M. Hirsch, W. Porod, and J.W.F. Valle, hep-ph/9907499; O. Haug, J.D. Vergados, A. Faessler, and S. Kovalenko, hep-ph/9909318; E.J. Chun and S.K. Kang, *Phys. Rev.* **D61**, 075012 (2000) (hep-ph/9909429).
18. K. Fukuura, T. Miura, E. Takasugi, and M. Yoshimura, Osaka Univ. preprint, OU-HET-326 (hep-ph/9909415).
19. G.K. Leontaris and J. Rizos, CERN-TH-99-268 (hep-ph/9909206); W. Büchmüller and T. Yanagida, *Phys. Lett.* **B445**, 399 (1999) (hep-ph/9810308); C.K. Chua, X.G. He, and W.Y. Hwang, hep-ph/9905340; A. Ghosal, hep-ph/9905470; J.E. Kim and J.S. Lee, hep-ph/9907452; U. Mahanta, hep-ph/9909518.
20. S.L. King, *Phys. Lett.* **B439**, 350 (1998) (hep-ph/9806440); S. Davidson and S.L. King, *Phys. Lett.* **B445**, 191 (1998) (hep-ph 9808296); E. Ma and D.P. Roy, *Phys. Rev.* **D59**, 097702 (1999) (hep-ph/9811266); Q. Shafi and Z. Tavartkiladze, *Phys. Lett.* **B451**, 129 (1999) (hep-ph/9901243); S. King, *Nucl. Phys.* **B562**, 57 (1999) (hep-ph/9904210); W. Grimus and H. Neufeld, hep-ph/9911465.

21. R. Barbieri, L.J. Hall, D. Smith, A. Strumia, and N. Weiner, *JHEP*, 9812:017 (1998) (hep-ph/9807235); R. Barbieri, L.J. Hall, and A. Strumia, *Phys. Lett.* **B445**, 407 (1999) (hep-ph/9808333); Y. Grossman, Y. Nir, and Y. Shadmi, *JHEP* 9810:007 (1998) (hep-ph/9808355); C.D. Froggatt, M. Gibson, and H.D. Nielson, *Phys. Lett.* **B446**, 256 (1999) (hep-ph/9811265).
22. C.H. Albright and S. Nandi, *Phys. Rev.* **D53**, 2699 (1996) (hep-ph/9507376); H. Nishiura, K. Matsuda, and T. Fukuyama, *Phys. Rev.* **D60**, 013006 (1999) (hep-ph/9902385); K.S. Babu, B. Dutta, and R.N. Mohapatra, *Phys. Lett.* **B458**, 93 (1999) (hep-ph/9904366).
23. M. Jezabek and Y. Sumino, *Phys. Lett.* **B440**, 327 (1998) (hep-ph/9807310); G. Altarelli and F. Feruglio, *Phys. Lett.* **B439**, 112 (1998) (hep-ph/9807353). G. Altarelli, F. Feruglio, and I. Masina, hep-ph/9907532.
24. B. Stech, *Phys. Lett.* **B465**, 219 (1999) (hep-ph/9905440); R. Dermisek and S. Raby, hep-ph/9911275; A. Aranda, C.D. Carone, and R.F. Lebed, hep-ph/0002044.
25. G.K. Leontaris, S. Lola, C. Scheich, J.D. Vergados, *Phys. Rev.* **D53**, 6381 (1996) (hep-ph/9509351); Y. Koide, *Mod. Phys. Lett.* **A11**, 2849 (1996) (hep-ph/9603376); P. Binetruy, S. Lavignac, S. Petcov, and P. Ramond, *Nucl. Phys.* **B496**, 3 (1997) (hep-ph/9610481); B.C. Allanach, *Phys. Lett.* **B450**, 182 (1999) (hep-ph/9806294); G. Eyal, *Phys. Lett.* **B441**, 191 (1998) (hep-ph/9807308); S. Lola and J.D. Vergados, *Prog. Part. Nucl. Phys.* **40**, 71 (1998) (hep-ph/9808269).
26. G. Costa and E. Lunghi, *Nuov. Cim.* **110A**, 549 (1997) (hep-ph/9709271).
27. M. Jezabek and Y. Sumino, *Phys. Lett.* **B440**, 327 (1998) (hep-ph/9807310).
28. G. Altarelli and F. Feruglio, *Phys. Lett.* **B439**, 112 (1998) (hep-ph/9807353); E.Kh. Akhmedov, G.C. Branco, and M.N. Rebelo, hep-ph/9911364.
29. M. Bando, T. Kugo, and K. Yoshioki, *Phys. Rev. Lett.* **80**, 3004 (1998) (hep-ph/9710417).

30. M. Abud, F. Buccella, D. Falcone, G. Ricciardi, and F. Tramontano, DSF-T-99-36 (hep-ph/9911238).
31. A.K. Ray and S. Sarkar, *Phys. Rev.* **D61**, 035007 (2000) (hep-ph/9908294).
32. J. Hashida, T. Morizumi, and A. Purwanto, hep-ph/9909208.
33. K. Oda, E. Takasugi, M. Tanaka, and M. Yoshimura, *Phys. Rev.* **D59**, 055001 (1999) (hep-ph/9808241).
34. Q. Shafi and Z. Tavartkiladze, BA-99-39 (hep-ph/9905202); D.P. Roy, Talk at 6th Topical Seminar on Neutrino and AstroParticle Physics, San Miniato, Italy, 17-21 May 1999 (hep-ph/9908262).
35. G. Altarelli, F. Feruglio, and I. Masina, hep-ph/9907532. See also R. Barbieri, P. Creminelli, and A. Romanino, *Nucl. Phys.* **B559**, 17 (1999) (hep-ph/9903460).
36. E. Malkawi, *Phys. Rev.* **D61**, 013006 (2000) (hep-ph/9810542); Y.L. Wu, *Eur. Phys. J.* **C10**, 491 (1999) (hep-ph/9901245).
37. For a review see H. Fritzsch, Talk at Ringberg Euroconference on New Trends in Neutrino Physics, Ringberg, Ger. 1998 (hep-ph/9807234), and references therein.
38. H. Fritzsch and Z.Z. Xing, *Phys. Lett.* **B372**, 265 (1996) (9509389).
39. M. Fukugita, M. Tanimoto, and T. Yanagida, *Phys. Rev.* **D57**, 4429 (1998) (hep-ph/9709388); M. Tanimoto, *Phys. Rev.* **D59**, 017304 (1999) (hep-ph/9807283); H. Fritzsch and Z.Z. Xing, *Phys. Lett.* **B440**, 313 (1998) (hep-ph/9808272); R.N. Mohapatra and S. Nussinov, *Phys. Lett.* **B441**, 299 (1998) (hep-ph/9808301); M. Fukugita, M. Tanimoto, and T. Yanagida, *Phys. Rev.* **D59**, 113016 (1999) (hep-ph/9809554); S.K. Kang and C.S. Kim, *Phys. Rev.* **D59**, 091302 (1999) (hep-ph/9811379); M. Tanimoto, T. Watari, and T. Yanagida, *Phys. Lett.* **B461**, 345 (1999) (hep-ph/9904338); M. Tanimoto, hep-ph/0001306. For an excellent review of flavor democracy schemes of neutrino mass mixing see H. Fritzsch and Z.Z. Xing, hep-ph/9912358.

40. G.C. Branco, M.N. Rebelo, and J.I. Silva-Marcos, *Phys. Lett.* **B428**, 136 (1998) (hep-ph/9802340); I.S. Sogami, H. Tanaka, and T. Shino-hara, *Prog. Theor. Phys.* **101**, 707 (1999) (hep-ph/9807449); G.C. Branco, M.N. Rebelo, and J.I. Silva-Marcos, hep-ph/9906368.
41. K.S. Babu and S.M. Barr, *Phys. Lett.* **B381**, 202 (1996) (hep-ph/9511446).
42. S.M. Barr, *Phys. Rev.* **D55**, 1659 (1997) (hep-ph/9607419).
43. J. Sato and T. Yanagida, *Phys. Lett.* **B430**, 127 (1998) (hep-ph/9710516).
44. C.H. Albright, K.S. Babu, and S.M. Barr, *Phys. Rev. Lett.* **81**, 1167 (1998) (hep-ph/9802314).
45. N. Irges, S. Lavignac, and P. Ramond, *Phys. Rev.* **D58**, 035003 (1998) (hep-ph/9802334); J.K. Elwood, N. Irges, and P. Ramond, *Phys. Rev. Lett.* **81**, 5064 (1998) (hep-ph/9807228).
46. Y. Nomura and T. Yanagida, *Phys. Rev.* **D59**, 017303 (1999) (hep-ph/9807325); N. Haba, *Phys. Rev.* **D59**, 035011 (1999) (hep-ph/9807552); G. Altarelli and F. Feruglio, *JHEP* 9811:021 (1998) (hep-ph/9809596); Z. Berezhiani and A. Rossi, *JHEP* 9903:002 (1999) (hep-ph/9811447); K. Hagiwara and N. Okamura, *Nucl. Phys.* **B548**, 60 (1999) (hep-ph/9811495); G. Altarelli and F. Feruglio, *Phys. Lett.* **B451**, 388 (1999) (hep-ph/9812475); K.S. Babu, J. Pati, and F. Wilczek, (hep-ph/9812538); M. Bando and T. Kugo, *Prog. Theor. Phys.* **101**, 1313 (1999) (hep-ph/9902204); Y. Nir and Y. Shadmi, *JHEP* 9905:023 (1999) (hep-ph/9902293); Y. Nomura and T. Sugimoto, hep-ph/9903334; K.I. Izawa, K. Kurosawa, Y. Nomura, and T. Yanagida, *Phys. Rev.* **D60**, 115016 (1999) (hep-ph/9904303); Q. Shafi and Z. Tavartkiladze, BA-99-63 (hep-ph/9910314); P. Frampton and A. Rasin, IFP-777-UNC (hep-ph/9910522).
47. R. Barbieri, L.J. Hall, G.L. Kane, and G.G. Ross, OUTP-9901-P (hep-ph/9901228); E. Ma, *Phys. Rev.* **D61**, 033012 (hep-ph/9909249).
48. C.H. Albright and S.M. Barr, *Phys. Lett.* **461**, 218 (1999) (hep-ph/9906296).

49. C.H. Albright and S.M. Barr, *Phys. Rev.* **D58**, 013002 (1998) (hep-ph/9712488).
50. H. Georgi and C. Jarlskog, *Phys. Lett.* **B86** (1979) 297.
51. S.M. Barr and S. Raby, *Phys. Rev. Lett.* **79**, 4748 (1998).
52. S. Dimopoulos and F. Wilczek, report No. NSF-ITP-82-07 (1981), in *The unity of fundamental interactions* Proceedings of the 19th Course of the International School of Subnuclear Physics, Erice, Italy, 1981 ed. A. Zichichi (Plenum Press, New York, 1983); K.S. Babu and S.M. Barr, *Phys. Rev.* **D48**, 5354 (1993); *Phys. Rev.* **D50**, 3529 (1994).
53. C.H. Albright and S.M. Barr, *Phys. Lett.* **B452**, 287 (1999) (hep-ph/9901318).
54. S. Weinberg, *Trans. NY Acad. Sci.* **38**, 185 (1977); F. Wilczek and A. Zee, *Phys. Lett.* **B70**, 418 (1977); H. Fritzsch, *Phys. Lett.* **B70**, 436 (1977).
55. C.H. Albright and S.M. Barr, hep-ph/0002155.
56. M. Tanimoto, *Phys. Rev* **D59**, 017304 (1999) (hep-ph/9807283).
57. M. Fukugita, M. Tanimoto, and T. Yanagida, *Phys. Rev.* **D57**, 4429 (1998) (hep-ph/9709388).
58. H. Fritzsch and Z.Z. Xing, *Phys. Lett.* **B440**, 313 (1998) (hep-ph/9808272); H. Fritzsch and Z.Z. Xing, hep-ph/9912358.
59. E.Kh. Akhmedov, G.C. Branco, and M.N. Rebelo, hep-ph/9912205 gives an analysis of U_{e3} that is different from but not inconsistent with the one given here.